Math 304 Midterm 1 Sample

Name: ______________________

This exam has 9 questions, for a total of 100 points.

Please answer each question in the space provided. You need to write **full solutions**. Answers without justification will not be graded. Cross out anything the grader should ignore and circle or box the final answer.

Question	Points	Score
1	18	
2	10	
3	10	
4	10	
5	10	
6	10	
7	16	
8	10	
9	6	
Total:	100	

Question 1. (18 pts)

Determine whether each of the following statements is true or false. You do NOT need to explain.

(a) Let V be a linear subspace of $\mathbb{R}^n$. We have vectors $v_1, \cdots, v_k$ and $w_1, \cdots, w_\ell$ in V. Suppose $v_1, \cdots, v_k$ are linearly independent, and $w_1, \cdots, w_\ell$ span V. Then $k \leq \ell$.

(b) Let v and w be two nonzero vectors in $\mathbb{R}^4$. Then v and w are linearly independent if only if v is not a scalar multiple of w.

(c) An $(n \times n)$ matrix is nonsingular if and only if it is row equivalent to the $(n \times n)$ identity matrix I_n.

(d) The following matrix

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is a row echelon form.

(e) Let A, B, C be three $(n \times n)$ square matrices. If $AB = AC$, then $B = C$.

(f) The linear system

$$\begin{aligned} x &+ 10y - 3z = 3 \\ 3x &+ 4y + 9z = 1 \\ 2x &+ 5y - 2z = 8 \end{aligned}$$

has exactly two solutions.

Solution:

(a) True

(b) True

(c) True

(d) False

(e) False

(f) False (a linear system can never have precisely two solutions.)

Question 2. (10 pts)

Solve the following linear system

$$\begin{cases} 2x_1 + 2x_2 - 3x_3 + x_4 + 13x_5 = 0 \\ x_1 + x_2 + x_3 + x_4 - x_5 = 0 \\ 3x_1 + 3x_2 - 5x_3 + 14x_5 = 0 \\ 6x_1 + 6x_2 - 2x_3 + 4x_4 + 16x_5 = 0 \end{cases}$$

Solution: The augmented matrix is

$$\left[\begin{array}{ccccc|c} 2 & 2 & -3 & 1 & 13 & 0 \\ 1 & 1 & 1 & 1 & -1 & 0 \\ 3 & 3 & -5 & 0 & 14 & 0 \\ 6 & 6 & -2 & 4 & 16 & 0 \end{array}\right]$$

Its row echelon form is

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & -4 & 0 \\ 0 & 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

So x_2 and x_5 are free variables. The solutions are

$$\begin{cases} x_1 = 2\beta - \alpha \\ x_2 = \alpha \\ x_3 = 4\beta \\ x_4 = -5\beta \\ x_5 = \beta \end{cases}$$

Question 3. (10 pts)

Compute the determinant of the following matrix

$$\begin{pmatrix} 2 & 1 & 2 & 1 \\ 3 & 0 & 1 & 1 \\ -1 & 2 & -2 & 1 \\ -3 & 2 & 3 & 1 \end{pmatrix}$$

Solution: use elementary row operations and keep track of how the determinant changes. (I skip the details here. You need to show the steps on the test.)

$$\begin{vmatrix} 2 & 1 & 2 & 1 \\ 3 & 0 & 1 & 1 \\ -1 & 2 & -2 & 1 \\ -3 & 2 & 3 & 1 \end{vmatrix} = 20$$

Question 4. (10 pts)
Determine whether the following matrix is nonsingular. If yes, find its inverse.

$$\begin{pmatrix} 1 & 2 & 3 & 1 \\ 3 & 7 & 7 & 4 \\ 2 & 4 & 5 & 1 \\ 2 & 4 & 4 & 1 \end{pmatrix}$$

Solution: form the following matrix

$$\left[\begin{array}{cccc|cccc} 1 & 2 & 3 & 1 & 1 & 0 & 0 & 0 \\ 3 & 7 & 7 & 4 & 0 & 1 & 0 & 0 \\ 2 & 4 & 5 & 1 & 0 & 0 & 1 & 0 \\ 2 & 4 & 4 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

apply elementary row operations to determine whether the matrix on the left is row equivalent to the identity matrix or not. If yes, then the resulting matrix on the right will be the inverse.

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 9 & -2 & -9 & 8 \\ 0 & 1 & 0 & 0 & -5 & 1 & 4 & -3 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 2 & 0 & -2 & 1 \end{array}\right]$$

So the inverse is

$$\begin{pmatrix} 9 & -2 & -9 & 8 \\ -5 & 1 & 4 & -3 \\ 0 & 0 & 1 & -1 \\ 2 & 0 & -2 & 1 \end{pmatrix}$$

Question 5. (10 pts)

Determine whether the following are subspaces.

(a) Let $\mathbb{P}_2$ be the vector space of all polynomials with degree equal to or less than 2. Is

$$W = \{p \in \mathbb{P}_2 \mid p(1) = 0\}$$

a subspace of $\mathbb{P}_2$?

Solution:

(1) Given two polynomials $p_1(x)$ and $p_2(x)$ in W, that is, $p_1(1) = 0$ and $p_2(1) = 0$. We have

$$(p_1 + p_2)(1) = p_1(1) + p_2(1) = 0.$$

So $(p_1 + p_2)(x)$ is in W.

(2) Given a polynomial $p(x)$ in W, that is, $p(1) = 0$, and $\alpha \in \mathbb{R}$. We have

$$(\alpha p)(1) = \alpha \cdot p(1) = 0.$$

So $(\alpha p)(x)$ is in W.

So W is a subspace of $\mathbb{P}_2$.

(b) Is $S = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}$ a subspace of $\mathbb{R}^3$?

Solution: S is the null space of the (1×3) matrix

$$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

So S is a subspace. Alternatively, we can verify that

(1) Given (x, y, z) and (a, b, c) in S, that is, $x + y + z = 0$ and $a + b + c = 0$, then

$$(x + a, y + b, z + c)$$

is a vector in S.

(2) Given a vector $(x, y, z) \in S$ and $\alpha \in \mathbb{R}$,

$$\alpha(x, y, z)$$

is a vector in S.

So S is a subspace of $\mathbb{R}^3$.

Question 6. (10 pts)

Determine whether $x = \begin{bmatrix} 4 \\ 5 \\ 6 \\ -1 \end{bmatrix}$ lies in the linear span of the vectors

$$v_1 = \begin{bmatrix} 1 \\ 3 \\ 2 \\ 5 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 4 \\ -1 \\ 2 \end{bmatrix} \text{ and } v_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 3 \end{bmatrix}.$$

Solution: Write down the matrix

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 3 & 4 & \text{-2} & 5 \\ 2 & \text{-1} & 1 & 6 \\ 5 & 2 & 3 & \text{-1} \end{array}\right]$$

by applying elementary row operations, we get

$$\left[\begin{array}{ccc|r} 1 & 0 & 0 & 7/3 \\ 0 & 1 & 0 & 1/3 \\ 0 & 0 & 1 & 5/3 \\ 0 & 0 & 0 & 55/12 \end{array}\right]$$

This is inconsistent. So no solution. In other words, x is not in the linear span of v_1, v_2 and v_3.

Question 7. (16 pts)

Determine whether

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, v_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \text{ and } v_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

form a basis of $\mathbb{R}^3$.

Solution: To verify a set of vectors is a basis or not, we need to determine two things.

(1) Are v_1, v_2, v_3 linearly independent or not? Consider the matrix

$$\begin{pmatrix} 1 & -2 & 1 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}$$

This is a nonsingular matrix. (You need to show the actual work here!) So v_1, v_2, v_3 are linearly independent.

(2) Do v_1, v_2, v_3 span $\mathbb{R}^3$? That is, we need to determine whether *every* vector in $\mathbb{R}^3$ can be written as a linear combination of v_1, v_2, v_3. Given any vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$, consider the linear system

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & a \\ 2 & 1 & 0 & b \\ 3 & 0 & 1 & c \end{array}\right]$$

Since $\begin{pmatrix} 1 & -2 & 1 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}$ is nonsingular, this linear system always have a unique solution. In other words, any vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ can be written as a linear combination of v_1, v_2, v_3. So v_1, v_2, v_3 span $\mathbb{R}^3$.

So we conclude that v_1, v_2, v_3 form a basis of $\mathbb{R}^3$.

Remark: There are alternative ways to solve this question. (See Theorem 3.4.4 of the textbook).

(a) For example, since we know the dimension of $\mathbb{R}^3$ is 3, once we show that the three vectors v_1, v_2, v_3 are linearly independent, this forces v_1, v_2, v_3 to span $\mathbb{R}^3$. In other words, in this particular question, we only need to show that v_1, v_2, v_3 are linearly independent. This is enough to conclude that v_1, v_2, v_3 form a basis.

(b) Yet another way, since we know the dimension of $\mathbb{R}^3$ is 3, once we show that the three vectors v_1, v_2, v_3 span $\mathbb{R}^3$, this forces v_1, v_2, v_3 to be linearly independent. In other words, in this particular question, we only need to show that v_1, v_2, v_3 span $\mathbb{R}^3$. This is enough to conclude that v_1, v_2, v_3 form a basis.

Question 8. (10 pts)

Let $\mathbb{P}_2$ be the vector space of all polynomials of degree equal to or less than 2. Determine whether the following polynomials in $\mathbb{P}_2$

$$p_1(t) = t - 1$$

$$p_2(t) = t + 1$$

$$p_3(t) = (t - 1)^2$$

are linearly independent or not.

Solution: There are various ways to solve this problem.

(1) First method: prove that $(t-1)$, $(t+1)$ and $(t-1)^2$ are linearly independent and span $\mathbb{P}_2(t)$.

(2) Second method: prove that $(t-1)$, $(t+1)$ and $(t-1)^2$ are linearly independent and use the fact $\dim \mathbb{P}_2(t) = 3$.

(3) third method: prove that $(t-1)$, $(t+1)$ and $(t-1)^2$ span $\mathbb{P}_2(t)$ and use the fact $\dim \mathbb{P}_2(t) = 3$.

Let us the second method. Consider a linear combination of $(t-1)$, $(t+1)$ and $(t-1)^2$ such that

$$a_1(t-1) + a_2(t+1) + a_3(t-1)^2 = 0.$$

Then we want to show that $a_1 = a_2 = a_3 = 0$ is the unique solution. This would imply that $(t-1)$, $(t+1)$ and $(t-1)^2$ are linearly independent.

Regroup the coefficients, and we have the following linear system:

$$\begin{cases} -a_1 + a_2 + a_3 = 0 \\ a_1 + a_2 - 2a_3 = 0 \\ a_3 = 0 \end{cases}$$

Solve this and indeed we have the unique solution $a_1 = a_2 = a_3 = 0$. So $(t-1)$, $(t+1)$ and $(t-1)^2$ are linearly independent.

Now we know that $\dim \mathbb{P}_2(t) = 3$. Then any 3 linearly independent vectors of $\mathbb{P}_2(t)$ form a basis. Therefore $(t-1)$, $(t+1)$ and $(t-1)^2$ form a basis.

Question 9. (6 pts)

Suppose A and B are nonsingular $(n \times n)$ matrices. We know that AB is also nonsingular. Show that

$$(AB)^{-1} = B^{-1}A^{-1}.$$

Solution: Notice that

$$B^{-1}A^{-1}(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = I$$

$$(AB)B^{-1}A^{-1} = A(BB^{-1})A^{-1} = A^{-1}IA = I$$

So by definition the inverse of AB, which is denoted by $(AB)^{-1}$, is $B^{-1}A^{-1}$. That is,

$$(AB)^{-1} = B^{-1}A^{-1}.$$